NAG Toolbox for MATLAB

f08qh

1 Purpose

f08qh solves the real quasi-triangular Sylvester matrix equation.

2 Syntax

```
[c, scale, info] = f08qh(trana, tranb, isgn, a, b, c, 'm', m, 'n', n)
```

3 Description

f08qh solves the real Sylvester matrix equation

$$\mathrm{op}(A)X \pm X\,\mathrm{op}(B) = \alpha C,$$

where $\mathrm{op}(A) = A$ or A^{T}, and the matrices A and B are upper quasi-triangular matrices in canonical Schur form (as returned by f08pe); α is a scale factor (≤ 1) determined by the function to avoid overflow in X; A is m by m and B is n by n while the right-hand side matrix C and the solution matrix X are both m by n. The matrix X is obtained by a straightforward process of back-substitution (see Golub and Van Loan 1996).

Note that the equation has a unique solution if and only if $\alpha_i \pm \beta_j \neq 0$, where $\{\alpha_i\}$ and $\{\beta_j\}$ are the eigenvalues of A and B respectively and the sign ($+$ or $-$) is the same as that used in the equation to be solved.

4 References

Golub G H and Van Loan C F 1996 *Matrix Computations* (3rd Edition) Johns Hopkins University Press, Baltimore

Higham N J 1992 Perturbation theory and backward error for $AX - XB = C$ *Numerical Analysis Report* University of Manchester

5 Parameters

5.1 Compulsory Input Parameters

1: **trana – string**

Specifies the option $\mathrm{op}(A)$.

trana = 'N'

$\mathrm{op}(A) = A$.

trana = 'T' or 'C'

$\mathrm{op}(A) = A^{\mathrm{T}}$.

Constraint: **trana** = 'N', 'T' or 'C'.

2: **tranb – string**

Specifies the option $\mathrm{op}(B)$.

tranb $=$ 'N'

op$(B) = B$.

tranb $=$ 'T' or 'C'

op$(B) = B^{\mathrm{T}}$.

Constraint: **tranb** $=$ 'N', 'T' or 'C'.

3: **isgn – int32 scalar**

Indicates the form of the Sylvester equation.

isgn $= +1$

The equation is of the form op$(A)X + X$ op$(B) = \alpha C$.

isgn $= -1$

The equation is of the form op$(A)X - X$ op$(B) = \alpha C$.

Constraint: **isgn** $= +1$ or -1.

4: **a(lda,$*$) – double array**

The first dimension of the array **a** must be at least max$(1,$ **m**$)$

The second dimension of the array must be at least max$(1,$ **m**$)$

The m by m upper quasi-triangular matrix A in canonical Schur form, as returned by f08pe.

5: **b(ldb,$*$) – double array**

The first dimension of the array **b** must be at least max$(1,$ **n**$)$

The second dimension of the array must be at least max$(1,$ **n**$)$

The n by n upper quasi-triangular matrix B in canonical Schur form, as returned by f08pe.

6: **c(ldc,$*$) – double array**

The first dimension of the array **c** must be at least max$(1,$ **m**$)$

The second dimension of the array must be at least max$(1,$ **n**$)$

The m by n right-hand side matrix C.

5.2 Optional Input Parameters

1: **m – int32 scalar**

Default: The first dimension of the array **a** and the second dimension of the arrays **a**, **c**. (An error is raised if these dimensions are not equal.)

m, the order of the matrix A, and the number of rows in the matrices X and C.

Constraint: **m** ≥ 0.

2: **n – int32 scalar**

Default: The first dimension of the array **b** and the second dimension of the array **b**. (An error is raised if these dimensions are not equal.)

n, the order of the matrix B, and the number of columns in the matrices X and C.

Constraint: **n** ≥ 0.

5.3 Input Parameters Omitted from the MATLAB Interface

lda, ldb, ldc

5.4 Output Parameters

1: **c(ldc,$*$) – double array**

The first dimension of the array **c** must be at least $\max(1,\mathbf{m})$

The second dimension of the array must be at least $\max(1,\mathbf{n})$

c contains the solution matrix X.

2: **scale – double scalar**

The value of the scale factor α.

3: **info – int32 scalar**

info $= 0$ unless the function detects an error (see Section 6).

6 Error Indicators and Warnings

info $= -i$

If **info** $= -i$, parameter i had an illegal value on entry. The parameters are numbered as follows:

1: **trana**, 2: **tranb**, 3: **isgn**, 4: **m**, 5: **n**, 6: **a**, 7: **lda**, 8: **b**, 9: **ldb**, 10: **c**, 11: **ldc**, 12: **scale**, 13: **info**.

It is possible that **info** refers to a parameter that is omitted from the MATLAB interface. This usually indicates that an error in one of the other input parameters has caused an incorrect value to be inferred.

info $= 1$

A and B have common or close eigenvalues, perturbed values of which were used to solve the equation.

7 Accuracy

Consider the equation $AX - XB = C$. (To apply the remarks to the equation $AX + XB = C$, simply replace B by $-B$.)

Let $\tilde{X}$ be the computed solution and R the residual matrix:

$$R = C - \left(A\tilde{X} - \tilde{X}B\right).$$

Then the residual is always small:

$$\|R\|_F = O(\epsilon)\left(\|A\|_F + \|B\|_F\right)\left\|\tilde{X}\right\|_F.$$

However, $\tilde{X}$ is **not** necessarily the exact solution of a slightly perturbed equation; in other words, the solution is not backwards stable.

For the forward error, the following bound holds:

$$\left\|\tilde{X} - X\right\|_F \leq \frac{\|R\|_F}{sep(A,B)}$$

but this may be a considerable overestimate. See Golub and Van Loan 1996 for a definition of $sep(A,B)$, and Higham 1992 for further details.

These remarks also apply to the solution of a general Sylvester equation, as described in Section 8.

8 Further Comments

The total number of floating-point operations is approximately $mn(m+n)$.

To solve the **general** real Sylvester equation

$$AX \pm XB = C$$

where A and B are general nonsymmetric matrices, A and B must first be reduced to Schur form (by calling f08pa, for example):

$$A = Q_1 \tilde{A} Q_1^{\mathrm{T}}$$

and

$$B = Q_2 \tilde{B} Q_2^{\mathrm{T}}$$

where $\tilde{A}$ and $\tilde{B}$ are upper quasi-triangular and Q_1 and Q_2 are orthogonal. The original equation may then be transformed to:

$$\tilde{A}\tilde{X} \pm \tilde{X}\tilde{B} = \tilde{C}$$

where $\tilde{X} = Q_1^{\mathrm{T}} X Q_2$ and $\tilde{C} = Q_1^{\mathrm{T}} C Q_2$. $\tilde{C}$ may be computed by matrix multiplication; f08qh may be used to solve the transformed equation; and the solution to the original equation can be obtained as $X = Q_1 \tilde{X} Q_2^{\mathrm{T}}$.

The complex analogue of this function is f08qv.

9 Example

```
trana = 'No transpose';
tranb = 'No transpose';
isgn = int32(1);
a = [0.1, 0.5, 0.68, -0.21;
     -0.5, 0.1, -0.24, 0.67;
     0, 0, 0.19, -0.35;
     0, 0, 0, -0.72];
b = [-0.99, -0.17, 0.39, 0.58;
     0, 0.48, -0.84, -0.15;
     0, 0, 0.75, 0.25;
     0, 0, -0.25, 0.75];
c = [0.63, -0.5600000000000001, 0.08, -0.23;
     -0.45, -0.31, 0.27, 1.21;
     0.2, -0.35, 0.41, 0.84;
     0.49, -0.05, -0.52, -0.08];
[cOut, scale, info] = f08qh(trana, tranb, isgn, a, b, c)
```

```
cOut =
   -0.4209    0.1764    0.2438   -0.9577
    0.5600   -0.8337   -0.7221    0.5386
   -0.1246   -0.3392    0.6221    0.8691
   -0.2865    0.4113    0.5535    0.3174
scale =
     1
info =
           0
```